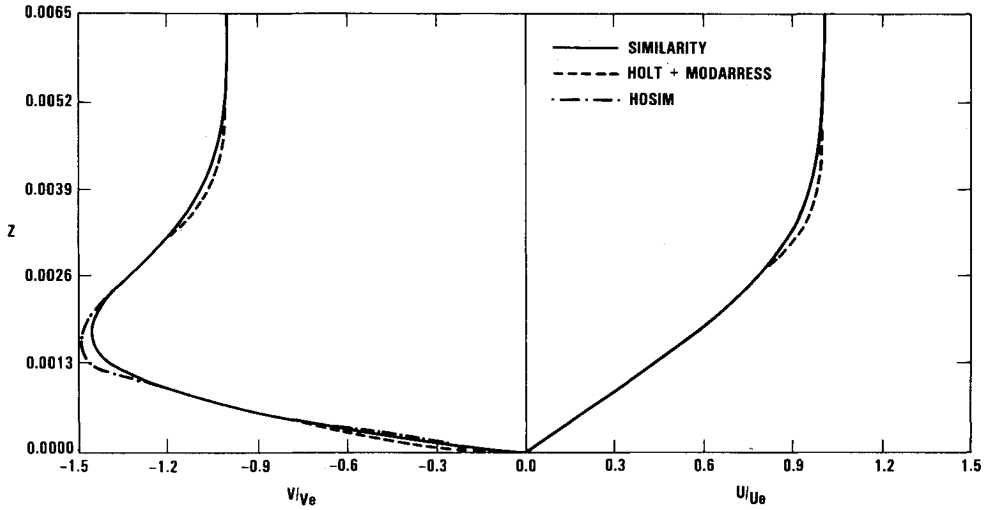

Fig. 2 Velocity profiles inside the boundary layer.

sidered. The external streamlines are given by the following expression.

$$U_e = \text{const}, \quad V_e = U_e(1 - CX) \tag{5}$$

where C is an arbitrary constant.

Loos has solved this problem using similarity considerations and has given the following expressions for velocities tangential and normal to the external streamline directions.

$$v_t = U_e\{1 + (1 - CX)^2\}^{-0.5}\{0.5f' + (1 - CX)(0.5f' - CXh)\}$$

$$v_n = U_e CX\{1 + (1 - CX)^2\}^{-0.5}\{h - 0.5f'\} \tag{6}$$

where f and h are given by the following differential equations and boundary conditions.

$$f''' + ff'' = 0, \quad h'' + fh' - 2f'h + 4 = 0 \tag{7}$$

with the boundary conditions,

$$f' = f = h = 0 \quad \text{when} \quad \eta = 0$$

$$f' \to 2 \text{ and } h \to 1 \quad \text{when} \quad \eta \to \infty \tag{8}$$

A higher order strip integral approach was applied to this problem. The initial values required for starting the solution were obtained from the above expression. A Runge-Kutta-Gill procedure with automatic step-size control was used for integrating the equations. A cubic spline interpolation technique was used to interpolate the velocities at strip boundaries.

Results

Figure 1 shows the comparison of results obtained by HOSIM and by the approach by Loos. The latter method can be considered as an exact method. The results show that the present approach has the capability to predict the boundary-layer quantities, even in the case of large cross flows. It can analyze flowfields involving a wide spectrum of boundary-layer profiles. Choice of a large number of strips would be similar to finite-difference analysis. But good results can be obtained by using a relatively small number of strips. The minimum number of strips required to produce reliable answers will be more for complex profiles compared to that for simple two-dimensional profiles. Five strips were used in this analysis, which produced the results shown in Fig. 1. Figure 2 shows the velocity profiles calulated by HOSIM plotted against the results by Holt and Modarress.[6] In both cases the initial values are obtained from the similarity solution. The X component of velocity was predicted almost exactly by HOSIM. The apparent greater accuracy of Fig. 2 is due to the reason that the initial conditions were applied at a location closer to the computational position than that for Fig. 1.

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A Flutter Eigenvalue Program Based on the Newton-Raphson Method

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Introduction

SINCE the elements of the matrix of the Laplace transformed equations of motion[1] for an elastic wing or aircraft are nonlinear functions of the transform parameter p ($= ik$, k being the reduced frequency), the eigenvalue problem associated with these equations is a nonlinear eigenvalue problem. In a program by Stark,[2] this problem was solved by iterative application of a routine for solution of a linear eigenvalue problem, while other authors[3,4] first introduced

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augmented states and then applied such a routine to the resulting larger system. Although extensive applications in practical flutter calculations show that the program of Ref. 2 is a useful tool, it is not sufficiently general for treatment of active control systems. A new approach has therefore been tried.

The new simple approach implies that the eigenvalues are calculated by solving the nonlinear equation $D(p)=0$ by iteration, where $D(p)$ is the determinant of the original system matrix $[Q_{m,n}]$ of the Laplace transformed equations. Combined with a new approximation[1] for calculation of the aerodynamic transfer functions, which are involved in $Q_{m,n}$, this approach has resulted in a new root-locus program which is briefly described in this Note.

Approximation of Transfer Functions

Aerodynamic transfer functions are related to indicial aerodynamic coefficients, which are defined[1] by special solutions to the boundary value problem for the perturbation velocity potential. Assuming that the wing deflection is a linear combination of given deflection modes $h_n(x,y)$ with undertermined coefficients $q_n(t)$ and using dimensionless quantities, we may write the boundary condition for the ordinary potential ϕ_n, which corresponds to h_n, as

$$\frac{\partial \phi_n}{\partial z} = \left(\frac{\partial h_n}{\partial x}\right) q_n + h_n \dot{q}_n \qquad \text{on the wing} \tag{1}$$

where $\dot{q}_n = \mathrm{d}q/\mathrm{d}t$. The coordinates x, y, z refer to some reference length L, the time t refers to L/U where U is the fligh speed, and ϕ_n to UL. The special solutions ϕ_n^1 and ϕ_n^2 satisfy

$$\begin{aligned} \frac{\partial \phi_n^r}{\partial z} &= \left(\frac{\partial h_n}{\partial x}\right) H(t) \qquad r=1 \\ & \qquad\qquad\qquad\qquad\qquad \text{on the wing} \\ &= h_n H(t) \qquad\qquad r=2 \end{aligned} \tag{2}$$

where $H(t)$ is the Heaviside unit step function. The generalized aerodynamic force coefficient $K_{m,n}(t)$ and the indicial coefficient $K_{m,n}^r(t)$ are dimensionless integrals of the products of h_m and pressure jumps determined by ϕ_n and ϕ_n^r via the linearized Bernoulli equation.

Since all relations are linear, it is seen that indicial coefficients defined in this way can be used for explicitly expressing aerodynamic forces for arbitrary motion in terms of the generalized coordinates $q_n(t)$. The relation reads

$$K_{m,n}(t) = \int_0^t K_{m,n}^1(t-\tau)\dot{q}_n(\tau)\mathrm{d}\tau + \int_{0-}^t K_{m,n}^2(t-\tau)\ddot{q}_n(\tau)\mathrm{d}\tau \tag{3}$$

By decomposing the indicial coefficients as

$$K_{m,n}^r(t) = K_{m,n}^r(\infty) - C_{m,n}^r(t) + D_{m,n}^r \delta(t) \tag{4}$$

where $\delta(t)$ is the Dirac delta function, and applying Laplace transformation to Eq. (3), we find that the aerodynamic transfer functions can be written

$$\begin{aligned} A_{m,n}(p) &= K_{m,n}^1(\infty) + [K_{m,n}^2(\infty) + D_{m,n}^1]p + D_{m,n}^2 p^2 \\ &\quad - [L\{C_{m,n}^1, p\} + pL\{C_{m,n}^2, p\}]p \end{aligned} \tag{5}$$

where $L\{C_{m,n}^r, p\}$ is the Laplace transform of the deficiency function $C_{m,n}^r(t)$. This is defined by Eq. (4), and $D_{m,n}^r$ is an apparent mass coefficient. The latter is nonzero only for incompressible flow.

In case of incompressible two-dimensional flow, all deficiency functions vary in the same way with time, and it can be shown that

$$C_{m,n}^r(t) = 2C_{m,n}^r(0)[1-\Phi(t)] \tag{6}$$

where $\Phi(t)$ is the Wagner function, and that

$$L\{C_{m,n}^r, p\} = 2C_{m,n}^r(0)[1-C(-ip)]/p \tag{7}$$

where $C(-ip)$ is Theodorsen's function. By inserting known values for $K_{m,n}^r(\infty)$, $D_{m,n}^r$, and $C_{m,n}^r(0)$ and Eq. (7) for the transform of the deficiency function into Eq. (5), we see that this reduces exactly to the formulas of the Küssner-Theodorsen theory. It may be noted that this derivation is very simple compared to the original one and yet more general; it is valid for all complex values of p for which the transform $L\{C_{m,n}^r, p\}$ exists.

Use of Cramer's rule for solution of the Laplace transformed equations of motion, application of the inversion theorem, and reduction of the integration path of the inverse Laplace transform integral show that the normal response function partly consists of the integral

$$I_{n,m} = \frac{-1}{2\pi i}\int_{-\infty+i0+}^{0+i0+} e^{pt}[F(p) - F(\bar{p})]\mathrm{d}p \tag{8}$$

where $\bar{p}$ is the complex conjugate of p and $F(p) = D^{n,m}/D$. $D^{n,m}$ is the cofactor of the element in the mth row and the nth column of D.

Since $C(-ip)$, which is discontinuous across $\arg(p)=\pi$, is involved in $F(p)$ in the case of incompressible two-dimensional flow, we see that $F(p)-F(\bar{p})$ and $I_{n,m}$ are nonzero in this case.

For three-dimensional compressible flow it is common practice to calculate $A_{m,n}(p)$ by numerical oscillating-surface methods, but simpler routines are required in flutter programs. Various approximations[5] have been proposed for this purpose, but all are not appropriate in all respects. Padé approximations[3,4] and expressions resulting from approximations of the Jones type to the Wagner function have poles instead of being discontinuous on $\arg(p)=\pi$. Therefore, so-called aerodynamic lag roots appear and yield contributions to the response.

Since $A_{m,n}(p)$ for three-dimensional flow must approach $A_{m,n}(p)$ for two-dimensional flow in a continuous way, an approximation of a different kind has been proposed[1] for subsonic flow.

This is based on an expression for the deficiency function of the form

$$C_{m,n}^r(t) = \sum_{k=k_i}^{k_s} B_{m,n}^{r,k}\left(\frac{a}{a+t}\right)^k \tag{9}$$

where k_i, k_s, a, and $B_{m,n}^{r,k}$ are arbitrary constants. Equation (9) is favorable, since the corresponding expression for $A_{m,n}(p)$ contains logarithmic terms which yield the desired discontinuity on $\arg(p)=\pi$. Furthermore, only one term may suffice for incompressible flow, because a result corresponding to the Garrick approximation for $\Phi(t)$ is obtained for $k_i=k_s=1$, $a=4$, $B_{m,n}^{r,1}=C_{m,n}^r(0)$, and an accurate approximation for typical modes of a rectangular aspect ratio 3 wing is obtained[1] for $k_i=k_s=3$, $a=5.5$, $B_{m,n}^{r,3}=C_{m,n}^r(0)$. For compressible flow, the deficiency function may have a peak at $t=0$ (which corresponds to the apparent mass term for incompressible flow), but it is nevertheless a regular function of $s=a/(a+t)$. Hence, it certainly can be approximated with sufficient accuracy for practical purposes by Eq. (10), which is a polynomial in s.

For supersonic flow, $C_{m,n}^r(t)$ becomes identically zero after a certain time, say, $t=a$ and is at least continuous for $0 \le t \le a$. We propose in this case an approximation of the form

$$C^r_{m,n}(t) = \sum_{k=k_i}^{k_s} B^{r,k}_{m,n}\left(1-\frac{t}{a}\right)^k \qquad 0 \le t \le a \tag{10}$$

but expect that several terms may be needed since the deficiency functions may be complicated for supersonic flow.

By inserting Eq. (9) or (10) into Eq. (5) we find

$$A_{m,n}(p) = A^0_{m,n} + A^1_{m,n}p + A^2_{m,n}p^2$$

$$-\sum_{k=k_i}^{k_s} (B^{1,k}_{m,n} + pB^{2,k}_{m,n})\, apF_k(ap) \tag{11}$$

where $A^0_{m,n} = K^1_{m,n}(\infty)$, $A^1_{m,n} = K^2_{m,n}(\infty) + D^1_{m,n}$, $A^2_{m,n} = D^2_{m,n}$,

$$F_k(p) = [1 - pF_{k-1}(p)]/(k-1) \qquad M<1$$

$$= [1 - kF_{k-1}(p)]/p \qquad M>1 \tag{12}$$

$$F_1(p) = e^p E_1(p) \qquad M<1$$

$$= [1-(1-e^{-p})/p]/p \qquad M>1 \tag{13}$$

$$E_1(p) = -\gamma - \ln(p) - \sum_{n=1}^{\infty} (-1)^n p^n/(n(n!)) \quad |\arg(p)| < \pi \tag{14}$$

and $\gamma = 0.577215\ldots =$ Euler's constant.

For a value of a chosen appropriately and constants $A^k_{m,n}$ and $B^{r,k}_{m,n}$ determined on the basis of discrete values of $A_{m,n}(p)$ (calculated by an oscillating-surface program), Eq. (11) may yield a useful approximation on the whole p plane. A comparison in Ref. 1 for incompressible flow supports this statement.

A Root-Locus Method

For obtaining the remaining and usually more important part of the normal response function or for determining the stability of the wing or aircraft, it is necessary to find the eigenvalues, i.e. the roots of $D(p) = 0$. This problem can be solved by the Newton-Raphson method, which gives the improved result $p_2 = p_1 - D(p_1)/D'(p_1)$ if a sufficiently close approximation p_1 to a root is given. Having determined the constants of Eq. (11), we may utilize this for calculation of $D(p_1)$ and $D'(p_1)$ for occurring values of p_1.

For large-order systems, it is not feasible, however, to form or use explicit expressions for $D(p)$ or $D'(p)$. Therefore, in the program developed, the determinant and its derivatives are calculated by efficient numerical methods; $D'(p)$ is obtained by a finite difference formula and $D(p)$ by a routine for solving linear equations. Since this routine, which generates $D(p)$ as a byproduct, is based on Gaussian elimination and triangularization, it is expected that the computer time will be acceptable not only for low-order systems.

The program requires approximate values for the roots as input, but this implies no problem since the roots shall be calculated for many increasing values of the flight speed. Suitable approximations for the first speed are furnished by eigenvalues for zero speed (which are usually determined by separate calculations or measurements) and those for a higher speed by the roots determined for the previous speed.

Results from a practical application for a subsonic aircraft based on ground vibration test data for deflection modes, natural frequencies, and generalized masses are illustrated in Fig. 1. The aerodynamic transfer function values that were used for determining the constants of Eq. (11) were calculated for four reduced frequencies for $M=0$ by a panel method for unsteady flow.

The figure shows that two roots for the same speed are not too close to each other for any speed, which is an experience also obtained from other examples. Therefore, problems due to coinciding or too closely located roots very seldom appear. The loci shown in the figure were obtained in a short continuous run for increasing Mach numbers (increased by 0.01 in each step) by doing three iterations for each root. This gave results in agreement with the earlier method,[2] but use of only one iteration in a second run produced nearly the same loci.

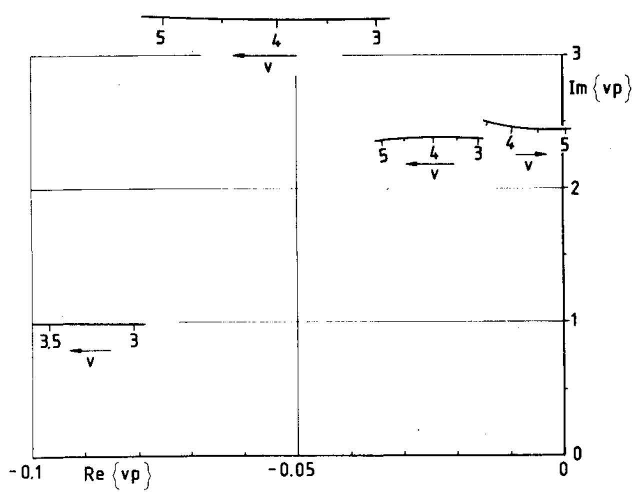

Fig. 1 Root loci for a passenger aircraft. $v = U/(\omega_r L)$ and $\omega_r/(2\pi)$ is a natural frequency.

Conclusions

A root-locus method, which is based on the Newton-Raphson method and on a new expression for approximating aerodynamic transfer functions, has been described. Since the determinant of the Laplace transformed equations of motion and its derivative are calculated by efficient numerical methods and as results from tests for low-order systems and incompressible flow are promising, it is expected that the method shall be useful also in more general cases.

Acknowledgment

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